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STEADY ROTATIONAL-OSCILLATORY MOTIONS IN A SYSTEM WHOSE UNPERTURBED MOTION IS STABLE

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Method of consecutive approximations is used to construct a rotational-oscillatory solution of a general system with a parameter, and its stability is studied on the basis of the well-known theorems of the First Liapunov Method. Earlier, analogously stated problems were investigated in connection with the periodic or oscillatory solutions in the system with small parameters.

We investigate a system whose general form is

$$dx_i / dt = F_i(t, x_1, \ldots, x_n, \lambda) \qquad (i = 1, \ldots, n) \tag{1}$$

where $t \in [t_0, \infty)$ is an independent variable and $\lambda \in [\lambda_1, \lambda_2]$ is a numerical parameter whose value is, in general, not small. We assume that real functions F_i satisfy the following conditions.

1) Functions F_i are defined for all $t \in [t_0, \infty)$, continuous and T-periodic where T is constant and independent of λ.

2) Functions F_i are periodic in $x_1, \ldots, x_p$ $(0 \leqslant p \leqslant n)$ with periods $T_1, \ldots, T_p$, respectively, the latter also independent of λ.

3) Functions F_i have partial derivatives of first and second order in $x_1, \ldots, x_p$ and λ satisfying the Lipschitz conditions, with constant independent of t in the vicinity of a point belonging to some region G of the variables x_i and $\lambda_0 \in [\lambda_1, \lambda_2]$ unbounded in the coordinates $x_1, \ldots, x_p$

We also assume that at a certain value of $\lambda = \lambda_0$ the system (1) admits, in G, an isolated solution of the form [1]

$$x_{i,0} = \varphi_i(\omega t) = \delta_i(T_i / 2\pi)\,\omega t + u_i(\omega t) \qquad (\omega = 2\pi / T) \tag{2}$$

where

$$\delta_i = 1 \quad (i \leqslant p), \qquad \delta_i = 0\ (i > p), \qquad u_i(\omega(t+T)) = u_i(\omega t)$$

Functions φ_i increase in such a manner, that the increment $\Delta x_{i,0} = \delta_i T_i$ they receive, is constant over any interval of time whose length is $\Delta t = T$. Solution (2) shall be called the rotational-oscillatory solution, and it shall be regarded as isolated provided that the system of equations in variations

$$d\xi_i / dt = p_{i1}\xi_1 + \ldots + p_{in}\xi_n \qquad (p_{ik}(\omega t) = (\partial F_i / \partial x_k)_0) \tag{3}$$

has no periodic solutions of period T. As we know [2], the sufficient condition for this to hold is, that no moduli of the roots of the characteristic equation for the system (3) are equal to unity. Poincaré and Liapunov methods were used in [2] to study the oscillatory systems of the type (3) with a small parameter.

We propose to construct, for all $t \in [t_0, \infty)$ an exact rotational-oscillatory solution of (1) and investigate its Liapunov stability for the values of λ sufficiently close to λ_0. When $\lambda = \lambda_0$, this solution will become (2) and will have the form

$$x_i(t, \lambda) = \varphi_i(\omega t) + (\lambda - \lambda_0)\, z_i(\omega t, \lambda) \quad (z_i(\omega(t+T), \lambda) \equiv z_i(\omega t, \lambda)) \tag{4}$$

To construct this solution we shall use the well-known method of consecutive approximations due to Malkin [2].

The principal result can then be stated in the form of the following theorem.

Theorem. If the solution (2) of the system (1) is isolated when $\lambda = \lambda_0$, then the basic perturbed system will admit, for λ sufficiently close to λ_0, one and only one solution of the form (4) belonging to G and reverting to its original form (2) when $\lambda = \lambda_0$.

This perturbed solution will be asymptotically stable when $|\lambda - \lambda_0|$ is sufficiently small, provided that the moduli of all roots of the characteristic equation in variations (3) are less than unity and unstable, when at least one of them is greater than unity.

Proof. Using the notation

$$\varepsilon = \lambda - \lambda_0, \qquad y_i = x_i - \varphi_i(\omega t)$$

we can represent the system (1) in the form

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_{ik} y_k + Y_i(t, y_1, \ldots, y_n) + \varepsilon \left(\frac{\partial F_i}{\partial \lambda}\right)_0 + \varepsilon f_i(t, y_1, \ldots, y_n, \varepsilon) \tag{5}$$

$$Y_i(t, y_1, \ldots, y_n) = F_i(t, \varphi_1 + y_1, \ldots, \varphi_n + y_n, \lambda_0) - (F_i)_0 - (p_{i1}y_1 + \ldots + p_{in}y_n)$$

$$f_i(t, y_1, \ldots, y_n, \varepsilon) = F_i^*(t, \varphi_1 + y_1, \ldots, \varphi_n + y_n, \varepsilon) - (\partial F_i / \partial \lambda)_0$$

$$\varepsilon F_i^*(t, x_1, \ldots, x_n, \varepsilon) = F_i(t, x_1, \ldots, x_n, \lambda) - F_i(t, x_1, \ldots, x_n, \lambda_0)$$

Since F_i are, by definition, smooth, the following estimates hold (see e.g. [2]) for y_i, y_i', y_i'', ε, ε' and ε'' with sufficiently small moduli:

$$|Y_i(t, y_1, \ldots, y_n)| < A \sum_{k=1}^{n} |y_k|^2 \tag{6}$$

$$|Y_i(t, y_1', \ldots, y_n') - Y_i(t, y_1'', \ldots, y_n'')| < B \sum_{j=1}^{n} (|y_j'| + |y_j''|) \sum_{k=1}^{n} |y_k' - y_k''|$$

$$|f_i(t, y_1', \ldots, y_n', \varepsilon') - f_i(t, y_1'', \ldots, y_n'', \varepsilon'')| < C \Big(\sum_{k=1}^{n} |y_k' - y_k''| + |\varepsilon' - \varepsilon''| \Big)$$

$$| f_i(t, y_1, \ldots, y_n, \varepsilon) | < C\Big(\sum_{k=1}^{n} | y_k | + | \varepsilon |\Big)$$

We easily see that the right hand sides of the system (5) are periodic in t, with the period equal to T. We construct its periodic solution using the method of consecutive approximations. The first approximation is defined from the system

$$\frac{dy_{i,1}}{dt} = \sum_{k=1}^{n} p_{ik}(\omega t)\, y_{k,1} + \varepsilon\left(\frac{\partial F_i}{\partial \lambda}\right)_0 \tag{7}$$

and any subsequent mth approximation is obtained as a periodic solution of the following general system

$$\frac{dy_{i,m+1}}{dt} = \sum_{k=1}^{n} p_{ik} y_{k,m+1} + Y_i(t, y_{1,m}, \ldots, y_{n,m}) + $$
$$+ \varepsilon\left(\frac{\partial F_i}{\partial \lambda}\right)_0 + \varepsilon f_i(t, y_{1,m}, \ldots, y_{n,m}, \varepsilon)$$

where the dependence on λ_0 is omitted from the expression to make it more compact. Since the variational system (3) has no periodic solutions of period T, then, by [2] the system (7) has a unique T-periodic solution satisfying the condition

$$\max | y_{i,1} | < \varepsilon D \max | (\partial F_i / \partial \lambda)_0 |$$

where the constant D depends on the coefficients p_{ik} but is independent of $(\partial F_i / \partial \lambda)_0$. Before all, we need to show that any qth approximation belongs to the admissible region and, that it is bounded. For this reason we assume that the inequalities

$$\max | y_{i,m} | < \varepsilon D \max | (\partial F_i / \partial \lambda)_0 | < \varepsilon N \quad (m = 1, \ldots, q \doteq 1) \tag{8}$$

hold, and then we prove that the qth approximation is also bounded by the quantity εN. Expressions (6) and (8) yield the following estimate for $y_{i,q}$

$$\max | y_{i,q} | < \varepsilon D \max | (\partial F_i / \partial \lambda)_0 | + \varepsilon^2 nADN^2 + \varepsilon^2 CD(nN + 1) \tag{9}$$

By a suitable choice of the small parameter ε, we can make the right hand side of (9) less than εN. This proves, that

$$\max | y_{i,m} | < \varepsilon N, \quad x_{i,m}(t, \lambda) \in G, \quad x_{i,m}(t, \lambda_0) = \varphi \quad (m = 1, \ldots, q, \ldots)$$

Next we shall show that the consecutive approximations converge uniformly. Let us find the differences $(y_{i,m+1} - y_{i,m})$ satisfying the following system:

$$\frac{d(y_{i,m+1} - y_{i,m})}{dt} = \sum_{k=1}^{n} y_{ik}(y_{k,m+1} - y_{k,m}) + [Y_i(t, y_{1,m}, \ldots, y_{n,m}) -$$
$$- Y_i(t, y_{1,m-1}, \ldots, y_{n,m-1})] + \varepsilon [f_i(t, y_{1,m}, \ldots, y_{n,m}, \varepsilon) - f_i[t, y_{1,m-1}, \ldots, y_{n,m-1}, \varepsilon)]$$

Putting $a_m = \max | y_{i,m} - y_{i,m-1} |$ we obtain

$$\max | y_{i,m+1} - y_{i,m} | < \varepsilon nD\,(2nBN + C)\, a_m$$

Assuming now that $a_{m+1} = \varepsilon nD(2nBN + C)a_m$, we see, that $a_m \to 0$ as $m \to \infty$, if $\varepsilon nD\,(2nBN + C) \leqslant 1$.

Thus, for sufficiently small ε the approximations $y_{i,m}$ converge uniformly to some periodic functions $y_i(\omega t, \varepsilon)$, which shall now be shown to satisfy (5). Let $y_i^*(\omega t, \varepsilon)$ be a unique T-periodic solution of the system

$$\frac{dy_i^*}{dt} = \sum_{k=1}^{n} p_{ik}(\omega t)\, y_k^* + Y_i(t, y_1(\omega t, \varepsilon), \ldots, y_n(\omega t, \varepsilon)) + \varepsilon (\partial F_i / \partial \lambda)_0 +$$
$$+ \varepsilon f_i(t, y_1(\omega t, \varepsilon), \ldots, y_n(\omega t, \varepsilon), \varepsilon)$$

Then, repeating the previous arguments we can construct the following estimates:

$$\max | y_i^* - y_{i,m} | < \varepsilon nD(2nBN + C) \max | y_i(\omega t, \varepsilon) - y_{i,m-1} |$$

Passing to the limit we can show that the functions $y_i(\omega t, \varepsilon)$ satisfy (5). To prove that, for a sufficiently small ε, the function $y_i(\omega t, \varepsilon)$ is a periodic solution of (5) unique in G, we assume that there exists another periodic solution $y_i'(\omega t, \varepsilon)$. Then the following estimate

$$\max | y_i' - y_i | < [\varepsilon nD(2nBN + C)]^l \max | y_i' - y_i |$$

where l is an arbitrary integer, will hold for $(y_i' - y_i)$. This implies that $y_i' \equiv y_i$.

To prove the theorem on the asymptotic stability of the constructed solution (4) for a sufficiently small ε, we make the usual substitution $x_i = x_i(\omega t, \lambda) + \xi_i$. Variations ξ_i satisfy the system

$$\frac{d\xi_i}{dt} = \sum_{k=1}^{n} [p_{ik}(\omega t) + f_{ik}(\omega t, \varepsilon)] \xi_k + R_i(\omega t, \xi_1, \ldots, \xi_n, \varepsilon)$$

in which the uniform estimates

$$| f_{ik} | \leqslant b | \varepsilon |, \qquad | R_i | \leqslant a | (| \xi_1 |^2 + \ldots + | \xi_n |^2)$$

hold for the functions R_i and f_{ik}.

Regarding the additions to (3) as "perturbations" with a sufficiently small Lipschitz constant, we can formulate, for sufficiently small ε and $| \xi_i |$ the following theorem due to Liapunov [2].

"If all characteristic indices of an unperturbed periodic motion have negative real parts, then the motion is asymptotically stable. If, however, at least one of these indices has a positive real part, then the unperturbed motion is unstable..."

Let $\rho_1, \ldots, \rho_n$ be the roots of the characteristic equation of (3). Since the characteristic indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ of this system are given by $\alpha_i = T^{-1} \ln \rho_i$, we can assume the theorem proved.

In conclusion we note, that a theorem analogous to that given above still holds, if the functions F_i have only first order partial derivatives in x_i and λ satisfying the Lipschitz conditions [2].

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